Coherent states and classical limits

1994 J. Phys. A: Math. Gen. 27 3257

(http://iopscience.iop.org/0305-4470/27/9/035)

J. Phys. A: Math. Gen. 27 (1994) 3257–3265. Printed in the UK

Coherent states and classical limits

Eckhard Meinrenken

Fakultät für Physik der Universität Freiburg, Hermann-Herder-Strasse 3, D-79104 Freiburg, Germany

Received 14 May 1993

Abstract. It is explained, using a coherent states approach that classical minimal coupling in the form introduced by Sternberg may be obtained as a classical limit from quantum minimal coupling. A similar procedure is used to exhibit Weinstein's symplectic structure on the space of WKB amplitudes as a classical limit.

1. Introduction

It is a well known exaggeration that quantum mechanics, described by a Hilbert space $\mathcal{H}$ and a self-adjoint Hamiltonian $\hat{H}$, may be considered to be a special case of classical mechanics. In fact, taking $\omega(v_1, v_2) = -\mathrm{Im}\,\langle v_1, v_2\rangle$ as a symplectic form on $\mathcal{H}$, the quantum time evolution is generated by the classical Hamiltonian $H(x) = \frac{1}{2}\langle x|\hat{H}|x\rangle$, and the commutator of two operators corresponds to the Poisson bracket of their respective Hamiltonians (see e.g. [1] p 460). Despite being rather useless for dealing with concrete problems, this may be one of the reasons why symplectic structures are so ubiquitous in physics: most reasonable physical theories are in some way linked to an underlying quantum theory, and sometimes the symplectic structure of the latter gives rise to a symplectic structure of the former.

This is exemplified most clearly in the theory of coherent states, where one is dealing with the embedding of some classical space as a submanifold of $\mathcal{H}$ or projective Hilbert space $P\mathcal{H}$. Consider, for instance, the case of the coherent states of a harmonic oscillator $\mathcal{H} = L^2\mathbb{R}^n$:

$$\langle x|q, p, \alpha\rangle_\hbar = \frac{1}{(\pi\hbar)^{n/4}} \exp\left(\frac{\mathrm{i}}{\hbar}(\alpha + x\cdot p) - \frac{1}{2\hbar}(x-q)\cdot(x-q)\right) \tag{1}$$

viewed as a map $(q, p, \alpha) \in T^*\mathbb{R}^n \times \mathbb{R} \to \mathcal{H}$. By pulling back the symplectic form of $\mathcal{H}$ one gets a presymplectic structure on $T^*\mathbb{R}^n \times \mathbb{R}$. Dividing out the null foliation just means projecting down to $T^*\mathbb{R}^n$, and the induced symplectic form on $T^*\mathbb{R}^n$ is just the canonical one. Moreover, the embedding is equivariant with respect to the natural action of the Heisenberg group $H(n) = \mathbb{R}^{2n} \times \mathbb{R}$, with its multiplication law

$$(q_1, p_1, \alpha_1)\cdot(q_2, p_2, \alpha_2) = (q_1 + q_2, p_1 + p_2, \alpha_1 + \alpha_2 - q_1\cdot p_2) \tag{2}$$

on $\mathcal{H}$ respectively $T^*\mathbb{R}^n \times \mathbb{R}$: The coherent states are generated from the simple Gaussian $|0, 0, 0\rangle$ as $|q, p, \alpha\rangle = U_\hbar(q, p, \alpha)|0, 0, 0\rangle$, where the representation $U_\hbar$ of $H(n)$ is given by the operators

$$U_\hbar(q, p, \alpha) = \exp[(\mathrm{i}/\hbar(\alpha + p\cdot x)]\exp(-q\partial/\partial x). \tag{3}$$

This aspect of the states $|q, p, \alpha\rangle$, being the orbit of some vector in $\mathcal{H}$ under an irreducible unitary representation of some 'coherence group', was used by Perelomov as the defining

0305-4470/94/093257+09$19.50

property for generalized coherent states. See his monograph [2] for a survey of the vast variety of applications of this concept, ranging from Lie group representation theory to laser physics.

The crucial property of the coherent states $|q, p, \alpha\rangle$ in the context of classical limits is that the expectation values of position $\hat{q}$ and momentum $\hat{p}$ are just q and p, with dispersion $\Delta q = \Delta p = \sqrt{\hbar} \to 0$. More generally, the expectation values of monomials in $\hat{q}$'s and $\hat{p}$'s factorize in leading order, e.g. $\langle \hat{q}^\alpha \hat{p}^\beta \rangle = \langle \hat{q} \rangle^\alpha \langle \hat{p} \rangle^\beta + \mathrm{O}(\hbar)$. In this sense, the coherent states become concentrated at points in the classical phase space, and a non-commutative algebra of quantum observables becomes a commutative algebra of classical observables.

2. Coherent states and the classical limit

In his article [3], Yaffe proposed a set of axioms for generalized coherent states abstracted from the above observations. We shall consider the following less restrictive but also less far reaching situation.

Let $\hbar$ be some real parameter varying in a subset of $\mathbb{R}_+$ having 0 as an accumulation point. Suppose we are given some Lie group G, with Lie algebra $\mathcal{G}$, and a family of unitary representations $\pi_\hbar$ of G on Hilbert spaces $\mathcal{H}_\hbar$. G is allowed to be infinite-dimensional, although we will be somewhat sloppy as far as technicalities are concerned. Denote by $\dot{\pi}_\hbar$ the corresponding representation of $\mathcal{G}$ on the space $\mathcal{H}_\hbar^\infty$ of C^∞ vectors† for $\pi_\hbar$. The universal enveloping algebra $U(\mathcal{G})$ of $\mathcal{G}$ will be taken as the algebra of observables; see e.g. Koch [4] or Landsman [5]. We continue to denote its representation on $\mathcal{H}_\hbar^\infty$ by $\dot{\pi}_\hbar$.

Example. Let M be a connected manifold and $\mathcal{H}_\hbar$ be the Hilbert space of square integrable half densities on M. Take G as the semidirect product $\mathrm{Diff}(M) \times_s \mathcal{F}(M)$ of the group of diffeomorphisms with the additive group of real-valued functions, with its product rule

$$(H_1, f_1) \circ (H_2, f_2) = (H_1 \circ H_2, f_1 + (H_1)_* f_2) \tag{4}$$

(the asterisk denotes push-forward), and consider the unitary representation

$$\pi_\hbar(H, f)(\phi_\hbar) = \exp\left(-\frac{\mathrm{i}}{\hbar} f\right) H_* \phi_\hbar. \tag{5}$$

This may be viewed as a generalization of the Heisenberg group $H(n)$. The Lie algebra $\mathcal{G}$ is the semidirect product $\mathcal{X}(M) \times_s \mathcal{F}(M)$ of vector fields on M with functions, with bracket

$$[(X_1, g_1), (X_2, g_2)] = ([X_1, X_2], X_1 g_2 - X_2 g_1). \tag{6}$$

Its representation reads

$$\dot{\pi}_\hbar(X, g)(\phi_\hbar) = \frac{1}{\mathrm{i}\hbar}\left(g + \frac{\hbar}{\mathrm{i}} \mathcal{L}_X\right) \phi_\hbar \tag{7}$$

whence its universal enveloping algebra gets represented by an algebra of differential operators.

Definition. A sequence $\phi_\hbar \in \mathcal{H}_\hbar^\infty$ of states with $||\phi_\hbar|| = 1$ will be called *coherent*, if the following axioms are fulfilled:

† If $\pi : G \to \mathcal{B}(\mathcal{H})$ is a unitary representation, a vector $\phi \in \mathcal{H}$ is called C^∞ if the function $g \mapsto \langle \psi | \pi(g) | \phi \rangle$ is in $C^\infty(G)$ for all $\psi \in \mathcal{H}$.

C.1. For all observables A in $U^k(\mathcal{G})$, the space of elements of degree k, the limit

$$\lim_{\hbar\to 0}(\mathrm{i}\hbar)^k \langle\phi_\hbar|\dot{\pi}(A)|\phi_\hbar\rangle =: \langle J^k(\phi), A\rangle \tag{8}$$

exists.

C.2. The dispersion vanishes in the $\hbar \to 0$ limit, in the sense that

$$\langle J^{k+l}(\phi), A\,B\rangle = \langle J^k(\phi), A\rangle\langle J^l(\phi), B\rangle \qquad \forall A \in U^k(\mathcal{G}), B \in U^l(\mathcal{G}). \tag{9}$$

Property C.1 implies that $J^k(\phi)$ is, in fact, a linear functional on $U^k(\mathcal{G})/U^{k-1}(\mathcal{G})$, and from C.2, it has already been determined by $J = J^1$. Note that if ϕ is a coherent sequence and $g \in G$, the transformed sequence $g\cdot\phi$ is also coherent, and the map J is Ad-equivariant:

$$J(\pi(g)\cdot\phi) = \mathrm{Ad}^*_{g^{-1}}(J(\phi)). \tag{10}$$

Two coherent sequences ϕ, ϕ' will be called equivalent if $J(\phi) = J(\phi')$. Let $[\phi]$ denote the equivalence class of ϕ and consider the induced mapping, also denoted J, from equivalence classes into $\mathcal{G}^*$. The orbit $\mathcal{Q} = G\cdot[\phi]$ which is to serve as 'our classical phase space' may thus be seen as a coadjoint orbit in $\mathcal{G}^*$.

Recall now that the representation $\pi_\hbar$ of G on the Hilbert space $\mathcal{H}_\hbar$ is a Hamiltonian group action, with moment map

$$I_\hbar : \mathcal{H}_\hbar \to \mathcal{G}^* \qquad \langle I_\hbar(\phi_\hbar), A\rangle = \frac{1}{2\mathrm{i}}\langle\phi_\hbar|\dot{\pi}_\hbar(A)|\phi_\hbar\rangle. \tag{11}$$

This map is equivariant with respect to the coadjoint action of G on $\mathcal{G}^*$; in particular it maps the G-orbit $G\cdot\phi_\hbar$ in $\mathcal{H}_\hbar$ to a coadjoint orbit in $\mathcal{G}^*$.

Let $\phi_\hbar \in \mathcal{H}_\hbar$,with $||\phi_\hbar|| = 1$, and $A_1, A_2 \in \mathcal{G}$. Then $\dot{\pi}_\hbar(A_i)\phi_\hbar$ may be regarded as tangent vectors to $P\mathcal{H}_\hbar$, and with the symplectic form $\omega_\hbar$ on $P\mathcal{H}_\hbar$ one has

$$\begin{aligned}\omega_\hbar(\dot{\pi}_\hbar(A_1)\phi_\hbar, \dot{\pi}_\hbar(A_2)\phi_\hbar) &= -\mathrm{Im}\langle\dot{\pi}_\hbar(A_1)\phi_\hbar, \dot{\pi}_\hbar(A_2)\phi_\hbar\rangle \\ &= \frac{1}{2\mathrm{i}}\langle\phi_\hbar|\dot{\pi}_\hbar([A_1, A_2])|\phi_\hbar\rangle \\ &= \langle I_\hbar(\phi_\hbar), [A, B]\rangle.\end{aligned} \tag{12}$$

Recall the Kirillov–Kostant–Souriau symplectic form of coadjoint orbits $\mathcal{O}$:

$$\omega_{\mathcal{O}}(z)(A_1\cdot z, A_2\cdot z) = \langle z, [A_1, A_2]\rangle \qquad \forall z \in \mathcal{O} \tag{13}$$

where $A_i\cdot z$ denotes the tangent vector at z corresponding to $A_i \in \mathcal{G}$. At least if G is finite-dimensional, equation (12) thus says that $I_\hbar$ is a symplectic reduction from $G\cdot\phi_\hbar$ onto the coadjoint orbit through $I_\hbar(\phi_\hbar)$. Moreover, condition (8) shows that, up to rescaling with $\hbar$, the family of coadjoint orbits $I_\hbar(G\cdot\phi_\hbar)$ tends towards the coadjoint orbit $J(G\cdot[\phi])$. In order to get the correct semiclassical limit, we are hence forced to let $\mathcal{Q} = G\cdot[\phi]$ be equipped with the symplectic form coming from its identification as a coadjoint orbit. The action of the coherence group G on $\mathcal{Q}$ will then be by symplectic transformations, and J becomes an Ad-equivariant moment map for this action. Finally, from (9), the mapping $\sigma : U^k(\mathcal{G}) \to \mathcal{F}(\mathcal{Q})$ dual to J^k represents the algebra of observables as a commutative algebra of functions on $\mathcal{Q}$. In the infinite-dimensional situation, problems may arise since the vectors $A\cdot z$ need not span the whole tangent space to $\mathcal{O}$ at z.

In the above example, a coherent sequence may be constructed as follows, imitating the harmonic oscillator coherent states. Choose some function $S \in C^\infty(M)$ such that, for some $x_0 \in M$,

$$\mathrm{Im}(S) \geqslant 0 \qquad \mathrm{Im}(S)(x) = 0 \Leftrightarrow x = x_0 \tag{14}$$

and such that the critical point of $\mathrm{Im}(S)$ at x_0 is non-degenerate. Associated with S is a distinguished covector based at x_0, namely $\xi_0 := \mathrm{dRe}(S)(x_0)$. Let ρ be a positive, compactly supported density on M such that $x_0 \in \mathrm{supp}(\rho)$, and consider the orbit through

$$\phi_\hbar := C_\hbar \exp[(\mathrm{i}/\hbar)S]\sqrt{\rho} \tag{15}$$

where $C_\hbar > 0$ is the normalizing constant. From the stationary phase theorem, the only contributions to integrals such as $\langle \phi_\hbar, \phi_\hbar \rangle$, $\langle \phi_\hbar | A | \phi_\hbar \rangle$ that are *not* exponentially small come from the point $x = x_0$. Moreover,

$$C_\hbar^2 \int \exp\Big(-\frac{2}{\hbar}\mathrm{Im}(S(x)\Big)b(x)\rho = b(x_0) + \mathrm{O}(\hbar)$$

for all smooth functions b on M. From this, it is not too hard to check properties C.1 and C.2, where

$$\langle J(\phi), (X, g)\rangle = g(x_0) + \langle \xi_0, X(x_0)\rangle \qquad \text{for all } X \in \mathcal{X}(M), g \in \mathcal{F}(M). \tag{16}$$

Since the action of the coherence group G transforms ϕ into a coherent state of a similar form, one gets,

$$\langle J((H, f)\cdot\phi), (X, g)\rangle = g(H(x_0)) + \langle (T_{x_0}H^{-1})^*(\xi_0) - \mathrm{d}f(H(x_0)), X(H(x_0))\rangle\rangle. \tag{17}$$

On the other hand,

$$(H, f)\cdot(x, \xi) = (H(x), (T_x H^{-1})^*(\xi) - \mathrm{d}f(H(x))) \tag{18}$$

is just the well known action of G on the cotangent bundle T^*M (see e.g. [6], p 220) and

$$\langle J_M(x, \xi), (X, g)\rangle = g(x) + \langle \xi, X(x)\rangle \tag{19}$$

its moment map. The G-action on T^*M is transitive since M is connected. This shows that $G \cdot [\phi]$ is a coherent system, parametrized by the points of $Q = T^*M$ considered as submanifold of $\mathcal{G}^*$. The map $\sigma : U(\mathcal{G}) \to \mathcal{F}(Q)$ is the usual principal symbol map for $\hbar$-differential operators (see e.g. [7, 8]).

3. Weinstein's symplectic structure on the 'isodrasts'

If one takes S to be real-valued in (15), the WKB wave $\phi_\hbar$ still satisfies C.1. Assuming without loss of generality that $\int \rho = 1$, $C_\hbar = 1$, one finds

$$\langle J^1([\phi]), (X, g)\rangle = \int_M \rho(g + X(S)) = \int_L \tilde{\rho}\langle J_M, (g, X)\rangle \tag{20}$$

where L is the graph of $\alpha := \mathrm{d}S : M \to T^*M$, i.e. a Lagrangian submanifold, and $\tilde{\rho} = \mathrm{d}S_*\rho$. More generally,

$$\langle J^k([\phi]), A\rangle = \int_L \tilde{\rho}\sigma(A) \tag{21}$$

where σ is the principal symbol for elements of $U(\mathcal{G})$ as defined in the last example. Although C.2 is clearly violated for these 'WKB wavefunctions', one can still try to use the construction from the previous section. Equation (21) shows that the equivalence class of $\phi_\hbar$ is completely determined by $\mathrm{d}S$ and ρ. Hence, Q will be the set of pairs $(L, \tilde{\rho})$ where L is the graph of an exaxt 1-form on M and $\tilde{\rho}$ a probability density on L. The action of $G = \mathrm{Diff}(M) \times_s \mathcal{F}(M)$ on Q is just the natural push-forward action coming from the symplectomorphism group of T^*M. A tangent vector to Q at $(L, \tilde{\rho})$ is a pair (β, Ψ), where β is an exact 1-form on M and Ψ is a density on L satisfying $\int_L \Psi = 0$. Equivalently,

$\beta = \mathrm{d}h$ may be identified with the vertical Hamiltonian vector field v whose Hamiltonian is the pull-back of h to T^*M. The corresponding flow is

$$\Lambda = \mathrm{graph}(\alpha) \to \Lambda_t = \mathrm{graph}(\alpha + t\beta) \qquad \rho \to \rho_t = \rho + t\Psi. \tag{22}$$

In particular, the tangent vector $(X, g)\cdot(L, \tilde{\rho})$ coming from the action of G is represented by the pair

$$(\beta, \Psi) = (\mathrm{d}(g + \langle\alpha, X\rangle), \widetilde{\mathcal{L}_X\rho}). \tag{23}$$

Suppose that $L = \mathrm{graph}(\mathrm{d}S)$, as above, and let $H_i = \langle J_M, (X_i, g_i)\rangle$ be the Hamiltonian corresponding to (X_i, g_i). According to (13), the symplectic form at $(L, \tilde{\rho})$ should be given by the formula

$$\begin{aligned}\omega((\beta_1, \Psi_1), (\beta_2, \Psi_2)) &= \omega\big((\mathrm{d}(g_1 + \langle\alpha, X_1\rangle), \Psi_1), (\mathrm{d}(g_2 + \langle\alpha, X_2\rangle), \Psi_2)\big)\\ &= \langle J^1([\phi]), [(X_1, g_1), (X_2, g_2)]\rangle\\ &= \int_M \Big(([X_1, X_2]S)(x) - (X_1g_2)(x) + (X_2g_1)(x)\Big)\rho\\ &= \int_M \big(X_1(X_2(S) + g_2) - X_2(X_1(S) + g_1)\big)\rho\\ &= -\int_L H_2\Psi_1 + \int_L H_1\Psi_2\end{aligned} \tag{24}$$

where the last step follows by a partial integration. Note that each observable $A \in U(\mathcal{G})$ gives, via its symbol $H = \sigma(A)$, rise to a Hamiltonian flow on Q, namely

$$\tilde{H}(L, \rho) = \int_L H\rho \tag{25}$$

and that the Hamiltonian flow of H *on* Q *is simply the flow induced by the flow of* $\sigma(A)$ on T^*M. In words, one gets Hamiltonian dynamics on the space of WKB waves.

The symplectic structure (24) is a special case of the following observation due to Weinstein [9]. Let (P, ω) be a symplectic manifold, and L be some fixed manifold with $\dim P = 2\dim L$. Consider the space $W\Lambda'(P)$ of all weighted Lagrangian immersions of L. A point of $W\Lambda'(P)$ is a pair (i, ρ) where $i : L \to P$ is Lagrangian, i.e. $i^*\omega = 0$, and ρ is a probability density on L. Any locally Hamiltonian vector field v on P and any density Ψ on L such that $\int_L \Psi = 0$ give a tangent vector at (i, ρ). Weinstein calls such a tangent vector 'isodrasic' if it corresponds to some globally Hamiltonian vector field $v = X_{\mathrm{H}}$. As it turns out, isodrastic tangent vectors define a foliation. Write down the following 2-form on the leaves of this foliation:

$$\tilde{\omega}((v_1, \Psi_1), (v_2, \Psi_2)) = \int_L i^*(\omega(v_1, v_2))\rho + \int_L H_1\Psi_2 - \int_L H_2\Psi_1. \tag{26}$$

The diffeomorphism group Diff(L) acts freely on $WF\Lambda'(P)$, thus making $W\Lambda'(P)$ a principal Diff(L) bundle over the space $W\Lambda(P) = W\Lambda'(P)/\mathrm{Diff}(L)$ of 'weighted Lagrangian submanifolds'. It can be shown that $\tilde{\omega}$ is invariant and horizontal with respect to the principal action, hence descends to a well defined 2-form on the isodrasts of $W\Lambda(P) = W\Lambda'(P)/\mathit{Diff}(L)$ *which turns out to be closed and non-degenerate.* In the discussion above, the immersion i was the embedding $\alpha = \mathrm{d}S$ of the zero section into T^*M, and since the Hamiltonians for v_i were constant along the cotangent fibration, $\omega(v_1, v_2) = \{H_1, H_2\} = 0$ whence the first term did not appear in (24).

The symplectomorphism group Smp(P) acts on weighted Lagrangian submanifolds by pushing them forward, and it is clear that this lifted action leaves the symplectic form $\tilde{\omega}$ invariant. Moreover, Hamiltonian vector fields on P give rise to Hamiltonian vector fields on $W\Lambda(P)$, with lifted Hamiltonian given by equation (25).

4. Sternberg's phase space as a coherent system

Our next example will be classical minimal coupling, as discovered by Sternberg [6, 10].

The basic data of Sternberg's model for the motion of a classical particle with internal degrees of freedom, coupled to a gauge field, are a principal bundle $\rho : P \to M$ with structure group K and an orbit $\mathcal{O}$ of the coadjoint action of K on $\mathcal{K}^*$. Recall that if X is a manifold on which K acts by diffeomorphisms, the space of orbits $P \times_K X := (P \times X)/K$ for the diagonal action $g \cdot (p, x) = (pg^{-1}, g \cdot x)$ is a smooth fibre bundle over M with typical fibre X. It is called an associated bundle and denoted $P \times_K X$.

If K is compact, the Borel–Weil–Bott theorem gives a 1:1 correspondence between *integral* coadjoint orbits and irreducible representations, so at least in this case there is some motivation to interpret $\mathcal{O}$ as a 'charge type'. Let $\omega_{\mathcal{O}}$ denote the symplectic structure (13) on $\mathcal{O}$ and recall that the embedding $J_{\mathcal{O}} : \mathcal{O} \to \mathcal{K}^*$ is a moment map for the K-action.

Sternberg's phase space is the associated bundle

$$\mathcal{O} \to Z := P^{\sharp} \times_K \mathcal{O} \to T^*M \tag{27}$$

where $P^{\sharp} \to T^*M$ denotes the pull-back of $P \to M$. The additional phase-space dimensions account for the internal degrees of freedom. Note that the fibres of Z are symplectic manifolds in a natural way. Using a connection and the corresponding vertical projection, one may extend their symplectic structure to obtain a 2-form $\tilde{\omega}_{\mathcal{O}}$ on Z. Adding the pull-back of the symplectic form ω_M on T^*M renders a non-degenerate 2-form on Z which, unfortunately, is not closed in general. It was discovered by Sternberg that if the connection comes from a principal connection (gauge field) on P, this can be remedied by adding a term involving the curvature:

$$\omega_Z = \omega_M + \tilde{\omega}_{\mathcal{O}} + \langle \tilde{J}_{\mathcal{O}}, F \rangle. \tag{28}$$

Here, $\tilde{J}_{\mathcal{O}} : P \times_K \mathcal{O} \to P \times_K \mathcal{K}^*$ is induced by the (equivariant) moment map J, the curvature F is considered as a 2-form on M with values in $P \times_K \mathcal{K}$, and some obvious pull-backs to Z have been omitted for convenience of notation. 'Minimal coupling' of a Hamiltonian defined on T^*M is achieved by pulling it back to Z. Hence, the gauge field enters the equations of motion via a modification of the symplectic structure.

In the case of electrodynamics, $K = U(1)$, the orbit $\mathcal{O}$ is just a point e (charge), the associated bundle can be identified with T^*M, and $\omega_Z = \omega_M + eF$, where F is the curvature (field strength) of the magnetic field.

We shall now show that, under the assumption that K is compact and $\mathcal{O}$ is integral, Sternberg's symplectic form may be derived by taking the classical limit of a corresponding quantum system.

Consider first the case where M is a point, so that Z is simply a coadjoint orbit $\mathcal{O}$. This situation was discussed in a similar context by Simon [11]. Let $U : K \times V \to V$ be an irreducible finite-dimensional unitary representation. Let λ be its maximal weight with respect to some choice of a maximal torus and $v \in V$ the maximal weight vector, $(v, v) = 1$, and let $[v]$ be the point in $P(V)$ corresponding to v. Recall (see e.g. [6]) that among the K-orbits in $P(V)$, there is a unique one that is a symplectic and complex (i.e. Kähler) submanifold of $P(V)$, and this is precisely the orbit $\mathcal{O}' = K \cdot [v]$. The moment map for this action is a symplectomorphism from $K \cdot [v]$ onto an integral coadjoint orbit $\mathcal{O}$, and this is just the orbit from which the representation is reconstructed via Borel–Weil–Bott.

Now, the irreducible representation with maximal weight $N\lambda$ ($N \in \mathbb{N}$) may be realized as a subrepresentation of $U^{\otimes N}$, and $v^{\otimes N}$ becomes a maximal weight vector. Denote this representation by $(U_\hbar, V_\hbar)$, where $\hbar^{-1} = N = 1, 2, \ldots$. Letting $\phi_\hbar := v^{\otimes N}$, one finds for $A_1, \ldots, A_k \in \mathcal{K}$:

$$\langle \phi_\hbar | \dot{U}_\hbar(A_1 \ldots A_k) | \phi_\hbar \rangle = N^k \langle v | A_1 | v \rangle \ldots \langle v | A_k | v \rangle + \mathrm{O}(N^{k-1}) \tag{29}$$

whence $\phi_\hbar$ renders a coherent system with

$$\langle J([\phi]), A \rangle = \mathrm{i} \langle v | U(A) | v \rangle \tag{30}$$

parametrized by the orbit $\mathcal{O}$.

For the general situation, we use a twisted version of the extremal cases where M is just a point and where the representation is trivial, i.e. where $Z = T^*M$. Form the associated bundles $E_\hbar := P \times_G V_\hbar$, along with their Hermitean fibre metrics $(\cdot, \cdot)$ inherited from the inner product on $V_\hbar$. One has an inner product on the space of $E_\hbar$-valued half densities on M:

$$\langle \zeta_1 \otimes \rho_1, \zeta_2 \otimes \rho_2 \rangle = \int_M (\zeta_1, \zeta_2) \overline{\rho_1} \rho_2. \tag{31}$$

Let

$$\mathcal{H}_\hbar := L^2 \big(E_\hbar \otimes |\Lambda|^{1/2} \big) \tag{32}$$

be *the completion of the space of square-integrable $E_\hbar$-valued half densities.* The automorphism group Aut(P), i.e. the group of all diffeomorphisms of P commuting with the K-action, acts on sections of $E_\hbar$ according to

$$H_*(\zeta) = H \circ \zeta \circ \check{H} \tag{33}$$

where $\check{H} \in \mathrm{Diff}(M)$ is the induced diffeomorphism of M, and on half-densities by push forward via $\check{H}$. With this action, we get a unitary representation $\pi_\hbar$ of the semidirect product $G = \mathrm{Aut}(P) \times_s \mathcal{F}(M)$ on $\mathcal{H}_\hbar$:

$$\pi_\hbar((H, f))\phi = \exp[-(\mathrm{i}/\hbar) f] H_* \phi. \tag{34}$$

Using a principal connection on P, the Lie algebra aut(P) of the automorphism group may be written as a product $\mathrm{aut}(P) = \mathrm{gau}(P) \times \mathcal{X}(M)$, corresponding to a decomposition into a vertical and horizontal part. Here, gau(P) is the Lie algebra of Gau(P), the group of all automorphisms of P inducing the identity on M. Elements of the gauge algebra gau(P) will be interpreted as sections of the associated bundle $P \times_K \mathcal{K}$ in what follows. An easy computation shows that the Lie bracket on $\mathcal{G} = (\mathrm{gau(P)} \times \mathcal{X}(M)) \times_s \mathcal{F}(M)$ is

$$\begin{aligned} &[(\sigma_1, X_1, g_1), (\sigma_2, X_2, g_2)] \\ &\quad = \big([\sigma_1, \sigma_2] + \nabla_{X_1} \sigma_2 - \nabla_{X_2} \sigma_1 + F(X_1, X_2) [X_1, X_2],\ X_1 g_2 - X_2 g_1 \big). \end{aligned} \tag{35}$$

The infinitesimal version of (34) reads

$$\dot{\pi}_\hbar(\sigma, X, g)\phi = -\frac{\mathrm{i}}{\hbar} \Big(g\phi + \frac{\hbar}{\mathrm{i}} \nabla_X \phi - \frac{\hbar}{\mathrm{i}} \sigma_* \phi \Big) \tag{36}$$

where $\nabla_X(\zeta \otimes \rho) := \nabla_X \zeta \otimes \rho + \zeta \otimes \mathcal{L}_X \rho$.

A coherent series for $\mathcal{H}_\hbar$ can be constructed as follows. Let W be the pre-image of $\mathcal{O}'$ in V. Choose a compactly supported section $\zeta \in \sec(P \times_K W)$ satisfying $(\zeta, \zeta)(x) \leqslant 1$, with equality only at a distinguished point $x = x_0$. Suppose, moreover, that the critical point of $-\log(\zeta, \zeta)$ at $x = x_0$ is non-degenerate. For $\hbar^{-1} = N = 1, 2, \ldots$, we now define

$$\phi_\hbar = C_\hbar \zeta^{\otimes N} \sqrt{\rho} \tag{37}$$

where ρ is a positive density on M not vanishing at x_0 and $C_\hbar$ is the normalizing constant.

The proof that this yields indeed a coherent system goes just like in the $L^2(M)$ situation. One has to study the asymptotics of

$$\int_M (\phi_\hbar, \pi_\hbar(\sigma, X, g)\phi_\hbar) \tag{38}$$

as $\hbar \to 0$. From

$$(\phi_\hbar, \dot{\pi}_\hbar(\sigma, X, g)\phi_\hbar) = \frac{N}{\mathrm{i}} \exp[N \log(\zeta, \zeta)] \left(\left(-\mathrm{i}\frac{(\zeta, \sigma_*\zeta)}{(\zeta, \zeta)} + \mathrm{i}\frac{(\zeta, \nabla_X\zeta)}{(\zeta, \zeta)} + g \right) \rho + \mathrm{O}(N^{-1}) \right)$$

one finds, using the stationary phase to determine the leading term in $\langle\phi_\hbar|(\sigma, X, g)|\phi_\hbar\rangle$ for $\hbar = N^{-1} \to 0$, that we must have

$$\langle J(\phi), (\sigma, X, g)\rangle = \langle J_M(x_0, \xi_0), (X, g)\rangle + \langle \tilde{J}_{\mathcal{O}}(z_0), \sigma\rangle.$$

Here, ξ_0 is the covector at x_0 defined by $\langle \xi_0, X(x_0)\rangle = -\mathrm{i}(\zeta, \nabla_X\zeta)(x_0)$ for all $X \in \mathcal{X}(M)$, and z_0 is the point of $P \times_K \mathcal{O} \subset P \times_K \mathcal{K}^*$ defined by $\langle z_0, \sigma(x_0)\rangle = -\mathrm{i}(\zeta, \sigma_*\zeta)(x_0)$ for all $\sigma \in \mathrm{gau}(P) = \mathrm{sec}(P \times_K \mathcal{K})$.

Moreover, the action of $(H, f) \in \mathrm{Aut}(P) \times_s \mathcal{F}(M)$ transforms $\phi_\hbar$ into a coherent series of a similar form. It hence follows that the G-orbit through $[\phi]$ is parametrized by the points of Sternberg's phase space $Z = P^\sharp \times_K \mathcal{O}$. According to section 2, the coherent state construction leads to a symplectic structure on Z, where the action of $\mathrm{Aut}(P) \times_s \mathcal{F}(M)$ becomes Hamiltonian and has J as its moment map. Note that the action of $\mathrm{Aut}(P)$ on Z depends on the choice of a connection on P. One observes, in particular, that the action of Gau(P) does not, in general, preserve the fibred structure of Z.

In fact, the action of $\mathrm{Aut}(P)$ on Z looks much simpler in the following description of Z as a symplectically reduced space, due to Weinstein [12]. Consider the left action of K on P given by $g \cdot p := pg^{-1}$. The lift of this action to T^*P is Hamiltonian, with Ad-equivariant moment map

$$\langle J_P^1(p, \pi), h\rangle = -\langle \pi, h_P(p)\rangle$$

where $h_P(p) = \partial/\partial t|_{t=0}(p \exp(th))$ is the fundamental vector field corresponding to $h \in \mathcal{K}$. The product $T^*P \times \mathcal{O}$ is hence a Hamiltonian K-space with moment map $J_P^1 + J_{\mathcal{O}}$. Form the reduced space

$$(T^*P)_{\mathcal{O}} = (J_P^1 + J_{\mathcal{O}})^{-1}(0)/K. \tag{39}$$

A connection on P induces a projection $\kappa : T^*P \to T^*M$ which is dual to the horizontal lifts $T_{\rho(p)}M \to T_pP$. Since κ is constant along the K-orbits it descends to a projection $(T^*P)_{\mathcal{O}} \to T^*M$ making $(T^*P)_{\mathcal{O}}$ into a symplectic fibre bundle. As it turns out, this fibre bundle is symplectically isomorphic to Z, the isomorphism being given by

$$(T^*P)_{\mathcal{O}} \to Z = P^\sharp \times_K \mathcal{O} \qquad K \cdot (p, \pi; q) \to K \cdot (\kappa(p, \pi), p; q). \tag{40}$$

Now, the semidirect product $\mathrm{Aut}(P) \times_s \mathcal{F}(M)$ acts on T^*P as a subgroup of $\mathrm{Diff}(P) \times_s \mathcal{F}(P)$ and has an Ad-equivariant moment map $J_P^2 : T^*P \to \left(\mathrm{aut}(P) \times_s \mathcal{F}(M)\right)^*$. It is readily checked that J_P^2 is constant along the K-orbits and J_P^1 is constant along the $\mathrm{Aut}(P) \times_s \mathcal{F}(M)$-orbits. The action therefore descends to a symplectic action on $(T^*P)_{\mathcal{O}} \cong Z$, with some equivariant moment map. We claim that this moment map is equal to J_Z above. Let σ' be the equivariant mapping $P \to \mathcal{K}$ corresponding to $\sigma \in \mathrm{gau}(P) = \mathrm{sec}(P \times_K \mathcal{K})$, equivalently viewed as an invariant vertical vector field on P.

The moment map for the G-action on T^*P reads

$$\langle J_P^2(p, \pi), (\sigma, X, g)\rangle = \langle \pi, \sigma(p)\rangle + \langle J_M(x, \xi), (X, g)\rangle$$

where $(x, \xi) = \kappa(p, \pi)$. But on $(J_P^1 + J_{\mathcal{O}})^{-1}(0)$,

$$\langle \pi, \sigma(p) \rangle = -\langle J_P^1(p, \pi), \tilde{\sigma}(p) \rangle = \langle J_{\mathcal{O}}(q), \tilde{\sigma}(p) \rangle = \langle \tilde{J}_{\mathcal{O}}(z), \sigma(x) \rangle.$$

To summarize:

Theorem 1. A connection on P gives rise to a natural Hamiltonian action of $\mathrm{Aut}(P) \times_s \mathcal{F}(M)$ on Sternberg's phase space $Z = P \times_K \mathcal{O}$, with equivariant moment map

$$\langle J_Z, (\sigma, X, g) \rangle = \langle J_M, (X, g) \rangle + \langle \tilde{J}_{\mathcal{O}}, \sigma \rangle. \tag{41}$$

The Hamiltonian vector field on Z corresponding to (σ, X, g) is given by the formula

$$(\sigma, X, g)_Z = \mathrm{Lift}(X_{T^*M} + g_{T^*M}) + \tilde{\sigma}_{\mathcal{O}} - \langle \tilde{J}_{\mathcal{O}}, i(X)F + \nabla\sigma \rangle^\sharp. \tag{42}$$

Here, 'Lift' is the horizontal lift with respect to the connection, $X_{T^*M} + g_{T^*M}$ is the Hamiltonian vector field on T^*M generated by (X, g), $\tilde{\sigma}_{\mathcal{O}}$ is the vertical vector field on Z induced by the infinitesimal action of $\sigma \in \mathrm{gau}(P)$, and $(\cdot)^\sharp$ is the map identifying 1-forms on T^*M with vector fields by means of the symplectic form.

Proof. It remains to prove (42). One has to show that $\iota(\sigma, X, g)_Z \omega_Z = \mathrm{d}\langle J_Z, (\sigma, X, g) \rangle$. By definition of ω_Z,

$$\begin{aligned} \iota(\mathrm{Lift}(X_{T^*M} + g_{T^*M}))\omega_Z &= \iota(X_{T^*M} + g_{T^*M})\omega_M + \langle \tilde{J}_M, \iota(X)F \rangle \\ &= \mathrm{d}\langle J_M, (X, g) \rangle + \langle \tilde{J}, \iota(X)F \rangle \\ \iota(\langle J, \iota(X)F \rangle^\sharp)\omega_Z &= \iota(\langle \tilde{J}, \iota(X)F \rangle)\omega_M \\ &= \langle \tilde{J}, \iota(X)F \rangle \end{aligned}$$

(since $\langle \tilde{J}, \iota(X)F \rangle^\sharp$ is tangent to the fibres of $P^\sharp \times_K \mathcal{O} \to P \times_K \mathcal{O}$). The vector field $\tilde{\sigma}_{\mathcal{O}}$ can be obtained by regarding $\langle \tilde{J}_{\mathcal{O}}, \sigma \rangle$ as a Hamiltonian on the fibres. Thus

$$\iota(\tilde{\sigma}_{\mathcal{O}})\omega_Z = \iota(\tilde{\sigma}_{\mathcal{O}})\omega_{\mathcal{O}}$$

is the vertical part of $\mathrm{d}\langle \tilde{J}_{\mathcal{O}}, \sigma \rangle$. On the other hand,

$$\iota(\langle \tilde{J}_{\mathcal{O}}, \nabla\sigma \rangle^\sharp)\omega_Z = \langle \tilde{J}_{\mathcal{O}}, \nabla\sigma \rangle$$

is its horizontal part. Collecting the terms finishes the proof. □

References

[1] Abraham R and Marsden J 1978 *Foundations of Mechanics* (New York: Addison-Wesley)

[2] Perelomov A 1972 Generalized coherent states *Commun. Math. Phys.* **26** 22

[3] Yaffe L 1982 Large *N* limits as classical mechanics *Rev. Mod. Phys.* **54** 2

[4] Koch M 1992 Quantum mechanics on homogeneous configuration spaces and classical particles in Yang–Mills fields *Preprint* Freiburg

[5] Landsman N P 1992 Induced representations, gauge fields, and quantization on homogeneous spaces *Rev. Math. Phys.* **4** 503–27

[6] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press) ch 3

[7] Guillemin V and Sternberg S 1977 Geometric asymptotics *Math. Surv.* **14**

[8] Meinrenken E 1992 *Rep. Math. Phys.* **31** 279–5

[9] Weinstein A 1990 Connections of Berry and Hannay type for moving Lagrangian manifolds *Adv. Math.* **82** 133–59

[10] Sternberg S 1977 On minimal coupling the symplectic mechanics of a classical particle in the presence of a Yang–Mills field *Proc. Acad. Sci.* **74** 5253–4

[11] Simon B 1980 The classical limit of quantum partition functions *Commun. Math. Phys.* **71** 247–76

[12] Weinstein A 1978 A universal phase space for particles in Yang–Mills fields *Lett. Math. Phys.* **2** 417–20